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ANALYSIS OF THE TSIEN NOZZLE

Prepared By

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February, 1963

BROWN ENGINEERING COMPANY INC.,
HUNTSVILLE, ALABAMA

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ANALYSIS OF THE TSIEN NOZZLE

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ABSTRACT

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A numerical method for designing a converging nozzle is presented. Since the calculations are tedious, an IBM 1401 FORTRAN program was developed and is included in the Appendix. A relation for the thrust coefficient in a converging nozzle is also derived for isentropic flow.

As an example of the application of the numerical method, the analysis of a specific Tsien nozzle is included.

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LIST OF SYMBOLS

A_t	Throat area
C_F	Thrust coefficient
K	Ratio of specific heat
M	Mach Number
P_a	Ambient Pressure
P_e	Stagnation Pressure
P_t	Exit Pressure
r	Radius
x	Co-ordinate
u	Velocity component in the x-direction
v	Velocity component in the r-direction
V	Velocity
w	Mass rate of flow
ψ	Stream function

FORTRAN SYMBOLS

RH(I)	Hermite Polynomial
PHI(I)	Derivative of the Velocity Function
FX(I)	Probability Integral
STRFU	Stream Function

INTRODUCTION

The analysis in this report is divided into two parts. The first part is the design of the contraction nozzle for a wind tunnel. The theory of this design is based on the paper, "On the Design of the Contraction Cone for a Wind Tunnel", which was published by Hsne-Shen Tsien. An IBM 1401 FORTRAN program which can be used to design the shape of nozzle is given in this report. The second part is to predict the thrust coefficient in the contraction nozzle with isentropic flow. An IBM 1401 FORTRAN program which can be used to predict the thrust coefficient for isentropic flow is also given in this report.

The purpose of this report is to provide theoretical information for a given Tsien nozzle to check the experimental results.

ANALYSIS

Analysis of the Tsien Nozzle

The following derivation is based on axi-symmetric, incompressible, irrotational, subsonic flow. The boundary conditions are assumed to be:

$$u = f_0(x) \qquad \text{at } r = 0 \tag{1-1}$$

$$v = 0 \qquad \text{at } r = 0 \tag{1-2}$$

The condition for irrotational flow is

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial r} = 0 \tag{1-3}$$

The equation of continuity is

$$\frac{\partial}{\partial x}(r u) + \frac{\partial}{\partial r}(r v) = 0 \tag{1-4}$$

By combining equations (1-2) and (1-3), the following equation can be obtained.

$$\frac{\partial u}{\partial r} = 0 \qquad \text{at } r = 0 \tag{1-5}$$

Hence, u is an even function.

By combining equations (1-2) and (1-4), the following equation can be obtained.

$$\frac{\partial}{\partial r}(r v) = 0 \qquad \text{at } r = 0$$

Hence v is an odd function.

The functions u and v can be written as

$$u = \sum_{n=0}^{\infty} r^{2n} f_{2n}(x) \tag{1-6}$$

$$v = \sum_{n=0}^{\infty} r^{2n+1} g_{2n+1}(x) \tag{1-7}$$

By substituting equations (1-6) and (1-7) into equation (1-3), one obtains:

$$\sum_{n=0}^{\infty} r^{2n+1} g'_{2n+1}(x) = \sum_{n=0}^{\infty} 2n\, r^{2n-1} f_{2n}(x)$$

Equating equal powers of r,

$$g'_{2n-1}(x) = 2n\, f_{2n}(x) \tag{1-8}$$

By substituting equations (1-6) and (1-7) into equation (1-4), one obtains

$$\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} r^{2n+1} f_{2n}(x)\right) + \frac{\partial}{\partial r}\left(\sum_{n=0}^{\infty} r^{2n+2} g_{2n+1}(x)\right) = 0$$

$$\sum_{n=0}^{\infty} r^{2n+1} f'_{2n}(x) + \sum_{n=0}^{\infty} (2n+2)\, r^{2n+1} g_{2n+1}(x) = 0$$

Equating equal powers of r,

$$f'_{2n}(x) = -(2n+2)\, g_{2n+1}(x)$$

Replacing n by (n - 1),

$$f'_{2(n-1)}(x) = -2n\, g_{2n-1}(x)$$

$$g_{2n-1}(x) = -\frac{1}{2n} f'_{2(n-1)}(x) \tag{1-9}$$

Equations (1-8) and (1-9) give the recurrence relation for the function f_{2n} as follows:

$$2n\, f_{2n}(x) = -\frac{1}{2n} f''_{2(n-1)}(x)$$

or

$$f_{2n}(x) = -\frac{1}{(2n)^2} f''_{2(n-1)} \tag{1-10}$$

Therefore,

$$f_{2n}(x) = \frac{(-1)^{n}}{(2)^{2n}\,(n!)^{2}}\, f_{o}^{(2n)}(x) \qquad (1\text{-}11)$$

Substituting these expressions back into equation (1-6), one obtains,

$$u = \sum_{n=0}^{\infty} \frac{(-1)^{n}\, r^{2n}}{2^{2n}\,(n!)^{2}}\, f_{o}^{2n}(x) \qquad (1\text{-}12)$$

By replacing n by (n + 1) in equation (1-9), the following expressions can be obtained.

$$g_{2n+1}(x) = \frac{-1}{2(n+1)}\, f'_{2n}(x)$$

$$= \frac{(-1)^{n}}{2(n+1)\; 2^{2n}\,(n!)^{2}}\; f_{o}^{2n+1}(x)$$

$$= \frac{(-1)^{n+1}\,(n+1)}{2^{2n+1}\;\left[(n+1)!\right]^{2}}\; f_{o}^{2n+1}(x)$$

Substituting these expressions back into equation (1-7), one obtains:

$$V = \sum_{n=0}^{\infty} \frac{r^{2n+1}\,(-1)^{n+1}\,(n+1)}{2^{2n+1}\;\left[(n+1)!\right]^{2}}\; f_{o}^{2n+1}(x)$$

Replacing n by (n - 1) in the preceding equation, the following expression is obtained.

$$V = \sum_{n=1}^{\infty} \frac{r^{2n-1}\,(-1)^{n}\, n}{2^{2n-1}\;(n!)^{2}}\; f_{o}^{2n-1}(x)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n}\, r^{2n-1}\, 2n}{2^{2n}\;(n!)^{2}}\; f_{o}^{2n-1}(x)$$

(1-13)

The resultant velocity is

$$V = (u^2 + v^2)^{\frac{1}{2}} \tag{1-14}$$

From the definition, a stream line can be defined as:

$$\psi(x,y) = \int_0^Y r\, u(x,y)\, dr \tag{1-15}$$

If the velocity distribution along the axis is known, the resultant velocity w and the stream lines can be calculated by using equations (1-14) and (1-15). The shape of the contraction cone is then determined by the streamline along withc the velocity still varies monotonically.

An IBM 1401 FORTRAN program for calculating the velocity and the streamline function is given in the appendix. The velocity distribution on the x-axis is assumed to be of the same form as the error function; i.e.,

$$u = f_o(x) = A + B\int_0^X \frac{1}{\sqrt{2\pi}}\, e^{-\left(\frac{x^2}{2}\right)} dx \tag{1-16}$$

The input data are the coefficients, i.e., A and B.

From the given contour (Figure 1), the continuity equation for incompressible flow was used to calculate the velocity distribution along the axis. The result is shown in the following table. Then, the velocity distribution is approximated by the following functions and plotted in Figure 2.

$$0 \leq X \leq 0.3342\ , \quad u = 0.5595 + 2.69849\int_0^X \frac{1}{\sqrt{2\pi}}\, e^{-\left(\frac{x^2}{2}\right)} dx$$

$$0.3342 \leq X \leq 0.8021, \quad u = 0.9127 + 0.30207\int_0^X \frac{1}{\sqrt{2\pi}}\, e^{\left(\frac{x^2}{2}\right)} dx$$

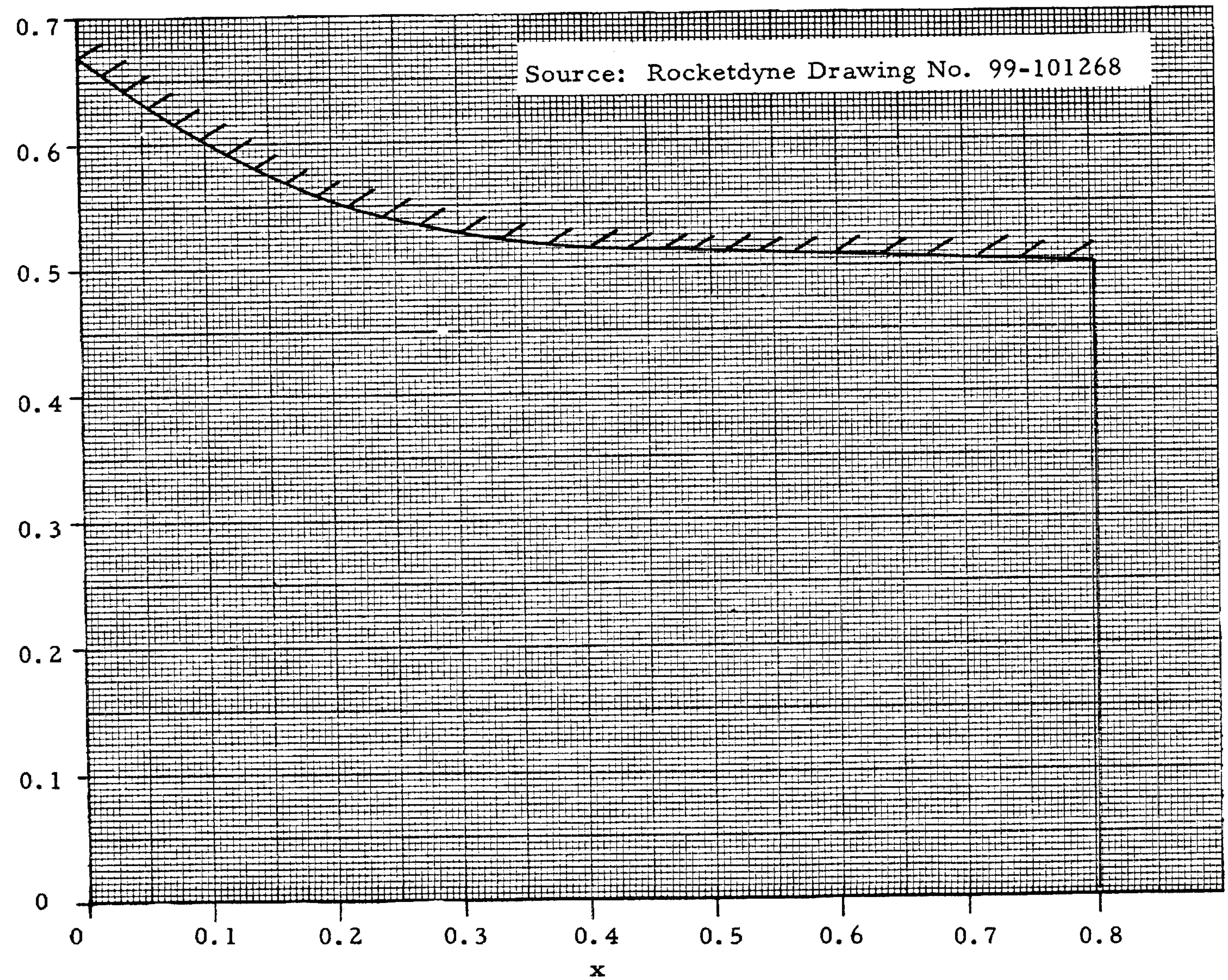


Figure 1 - Internal Contour of a Tsien Nozzle

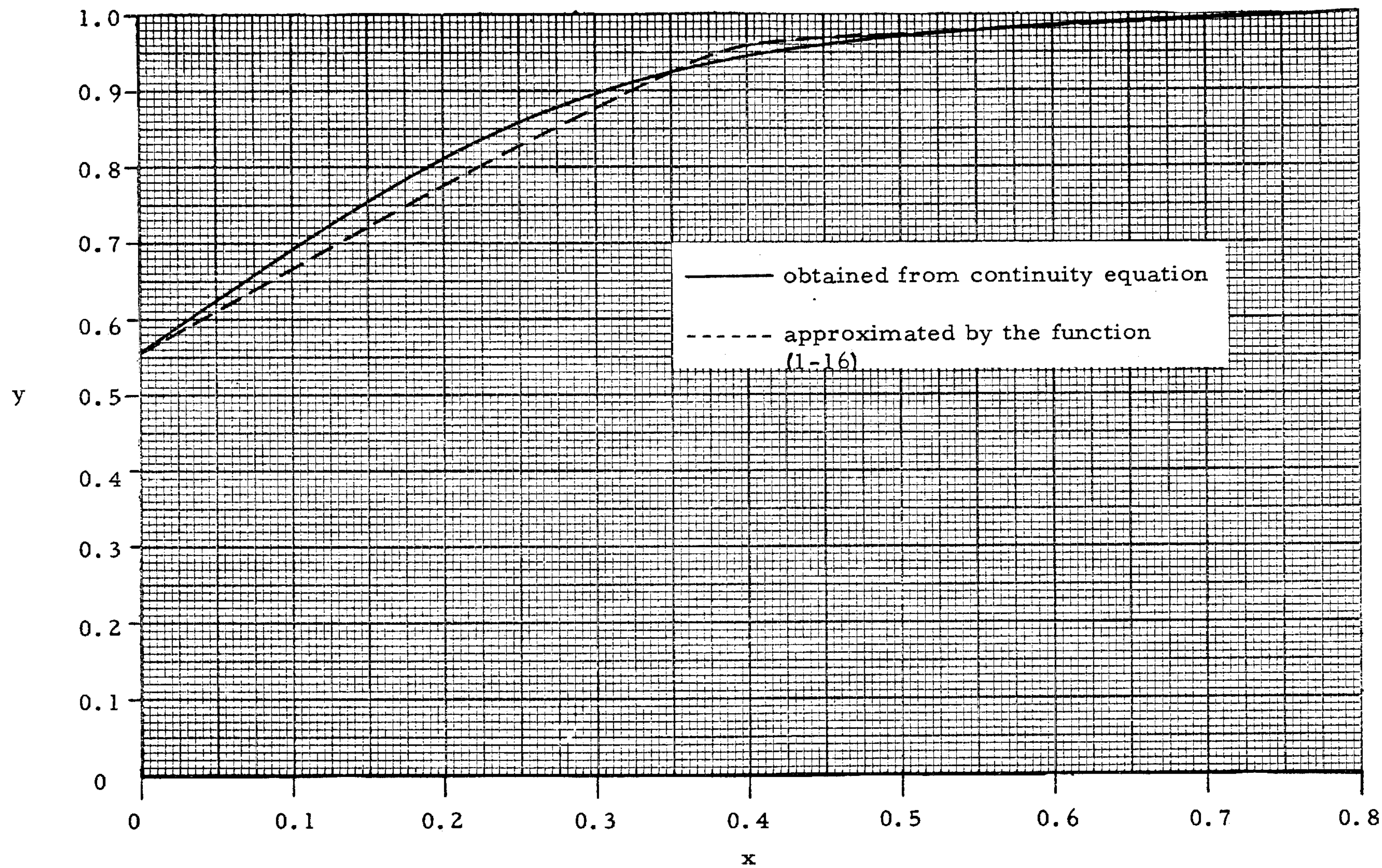


Figure 2 - Velocity Distribution on the X-Axis

TABLE I
TSIEN NOZZLE

x	y	y^2	$u_{n-1} = \frac{\pi y_n^2}{\pi y_{n-1}^2} u_n$
.8021	.500	.2500	1.0000
.7353	.5007	.2507	.9972
.6684	.5020	.2520	.9920
.6016	.5040	.2540	.9842
.5348	.5060	.2560	.9765
.4679	.5100	.2601	.9611
.4011	.5154	.2656	.9412
.3342	.5234	.2739	.9127
.2674	.5354	.2866	.8723
.2005	.5548	.3078	.8122
.1337	.5836	.3406	.7340
.0668	.6217	.3865	.6468
.0000	.6684	.4468	.5595

The Determination of C_F vs. P_e/P_a for the Given Contour of the Tsien Nozzle

The flow in the nozzle is assumed to be isentropic. Applying the momentum equation to the control volume, the relation

$$F = w V_t + (P_t - P_a) A_t \tag{2-1}$$

is obtained.

Equation (2-1) may be written in a dimensionless form as

$$\frac{F_o}{P_e A_t} = \frac{W}{P_e A_t} V_t + \left(\frac{P_t}{P_e} - \frac{P_a}{P_e} \right) \tag{2-2}$$

Assume the flow to be a perfect gas.

$$\frac{w}{A} = \rho V = \frac{P}{RT} V = \frac{PV}{\sqrt{KRT}} \sqrt{\frac{K}{R}} \sqrt{\frac{T_o}{T}} \frac{1}{\sqrt{T_o}} \tag{2-3}$$

Since

$$M = \frac{V}{\sqrt{KRT}}$$

$$\frac{T_e}{T} = 1 + \frac{K-1}{2} M^2$$

(Stagnation temperature ratio substituting the relations for M and T_e/T into 2-3), the following relationship is obtained.

$$\frac{w}{A} = \sqrt{\frac{K}{R}} \sqrt{\frac{P}{T_e}} M \sqrt{1 + \frac{K-1}{2} M^2} \tag{2-4}$$

(2-5)

By using equation (2-5), P in equation (2-4) can be eliminated.

$$\frac{P_e}{P} = \left(1 + \frac{K-1}{2} M^2\right)^{\frac{K}{K-1}} \tag{2-6}$$

w/A is maximum when M = 1.

$$\frac{W}{A_t P_e} = \sqrt{\frac{K}{R}\left(\frac{2}{K+1}\right)^{\frac{K+1}{K-1}}} \frac{1}{\sqrt{T_e}} \tag{2-7}$$

From the energy equation and isentropic law, the exit velocity can be written as

$$V_t = \sqrt{2 C_p (T_e - T_t)} = \sqrt{2 C_p T_e} \sqrt{1 - \frac{T_t}{T_e}}$$

$$= (2 C_p T_e)^{\frac{1}{2}} \left(1 - \left(\frac{P_t}{P_e}\right)^{\frac{K-1}{K}}\right)^{\frac{1}{2}} \tag{2-8}$$

Substituting (2-7) and (2-8) into (2-2), one obtains

$$C_F = \frac{F_o}{P_e A_t} = K\sqrt{\frac{2}{K-1}\left(\frac{2}{K+1}\right)^{\frac{K+1}{K-1}}} \sqrt{1 - \left(\frac{P_t}{P_e}\right)^{\frac{K-1}{K}}} + \left(\frac{P_t}{P_e} - \frac{P_a}{P_e}\right) \tag{2-9}$$

Since the nozzle is a simple converging nozzle, ($A_e = A_t$), P_t/P_e is the critical pressure ratio.

$$\frac{P_t}{P_e} = \left(\frac{2}{K+1}\right)^{\frac{K}{K-1}} \tag{2-10}$$

Substituting equation (2-10) into equation (2-9), one obtains:

$$\frac{F}{P_o A_t} = K\sqrt{\frac{2}{K-1}\left(\frac{2}{K+1}\right)^{\frac{K+1}{K-1}}}\sqrt{1-\left(\frac{2}{K+1}\right)} + \left(\frac{2}{K+1}\right)^{\frac{K}{K-1}} - \frac{P_a}{P_e}$$

$$= K\sqrt{\frac{2}{K-1}\left(\frac{2}{K+1}\right)^{\frac{K+1}{K-1}}}\sqrt{\frac{K-1}{K+1}} + \left(\frac{2}{K+1}\right)^{\frac{K}{K-1}} - \frac{P_a}{P_e}$$

$$= K\sqrt{\frac{2}{K+1}\left(\frac{2}{K+1}\right)^{\frac{K+1}{K-1}}} + \left(\frac{2}{K+1}\right)^{\frac{K}{K-1}} - \frac{P_a}{P_e}$$

$$= K\sqrt{\frac{2}{K+1}\left(\frac{2}{K+1}\right)^{\frac{K}{K-1}}\left(\frac{2}{K+1}\right)^{\frac{1}{K-1}}} + \left(\frac{2}{K+1}\right)^{\frac{K}{K-1}} - \frac{P_a}{P_e}$$

$$= K\left(\frac{2}{K+1}\right)^{\frac{K}{K-1}} + \left(\frac{2}{K+1}\right)^{\frac{K}{K-1}} - \frac{P_a}{P_e}$$

$$= \left(\frac{2}{K+1}\right)^{\frac{K}{K-1}}(K+1) - \frac{P_a}{P_e}$$

$$= 2\left(\frac{2}{K+1}\right)^{\frac{1}{K-1}} - \frac{P_a}{P_e} \qquad (2\text{-}11)$$

From Equation (2-11) it is shown that the thrust coefficient in a simple converging nozzle is a function of the ratio of the specific heat and pressure ratio, P_e/P_a.

DISCUSSION

The following discussion is divided into two parts. The first part deals with the shape of the given Tsien Nozzle, and the second part deals with the performance of the given nozzle, i.e., C_F vs. P_c/P_a.

(1) Since the shape of the Tsien nozzle is known and the flow is incompressible, the continuity equation may be used to calculate the velocity distribution along the x-axis. Then the velocity distribution as calculated from the continuity relation is approximated by the function, (1-16). Since the velocity distribution on the x-axis is known, the local velocity and stream lines can be calculated by the program described in the Appendix. The result of these calculations is shown in Figure 3. Any one of the stream lines along which the local velocities are less than unity can be considered to be the contour of a convergent nozzle. The given shape of the Tsien nozzle is also plotted in Figure 3. The local velocity is subsonic throughout the nozzle and the streamlines are approximately tangent to the original contour. This Tsien nozzle can be considered a typical Tsien nozzle.

According to Tsien's design criteria, boundary layer separation and an adverse pressure gradient may be avoided if the pressure along the wall decreases monotonically. The wind tunnel test results for this Tsien nozzle (plotted in Figure 4) shows that the pressure does decrease

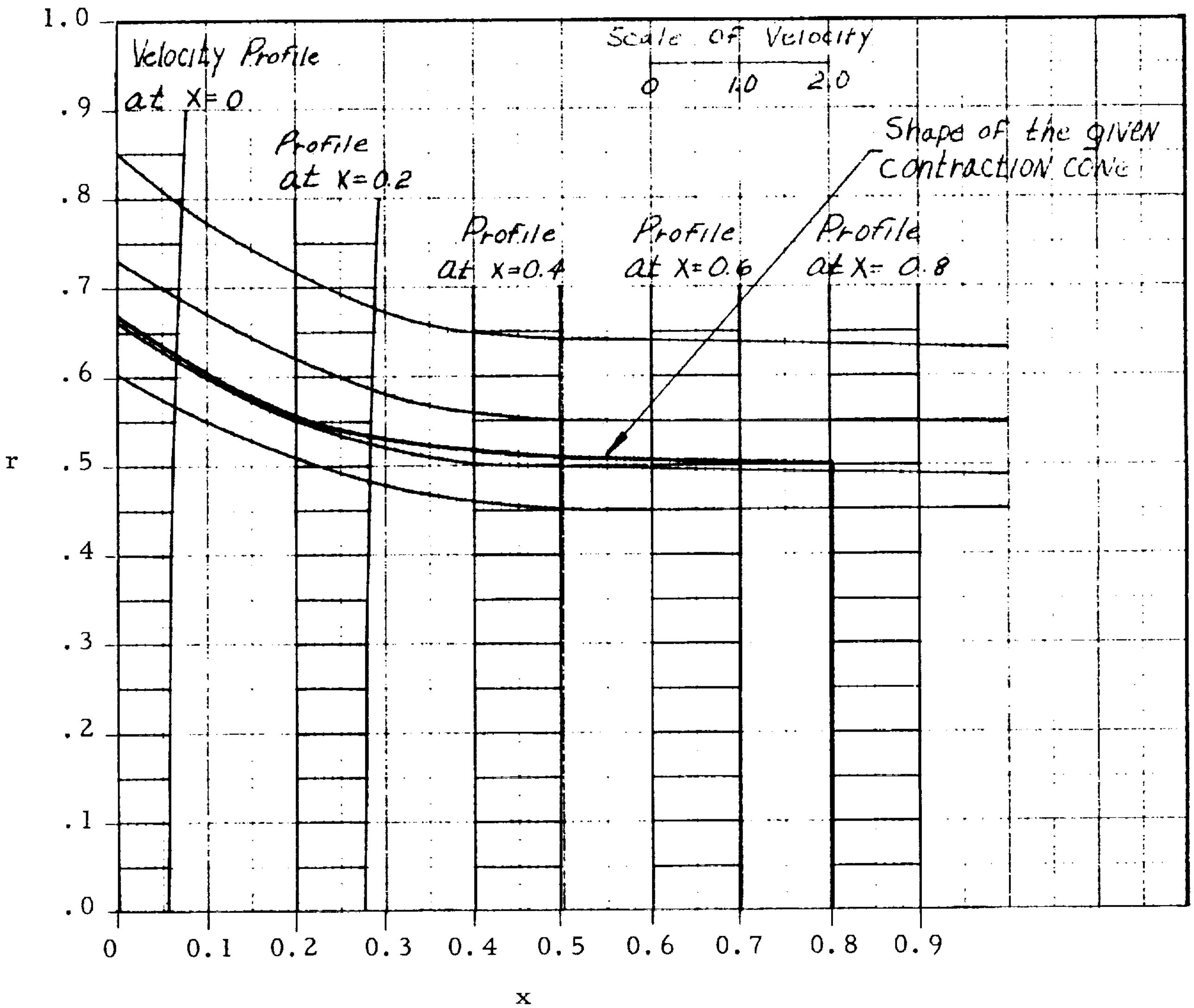


Figure 3 - Stream Lines in the x, r Plane of the Contraction Cone Together with the Velocity Profiles of Different Sections

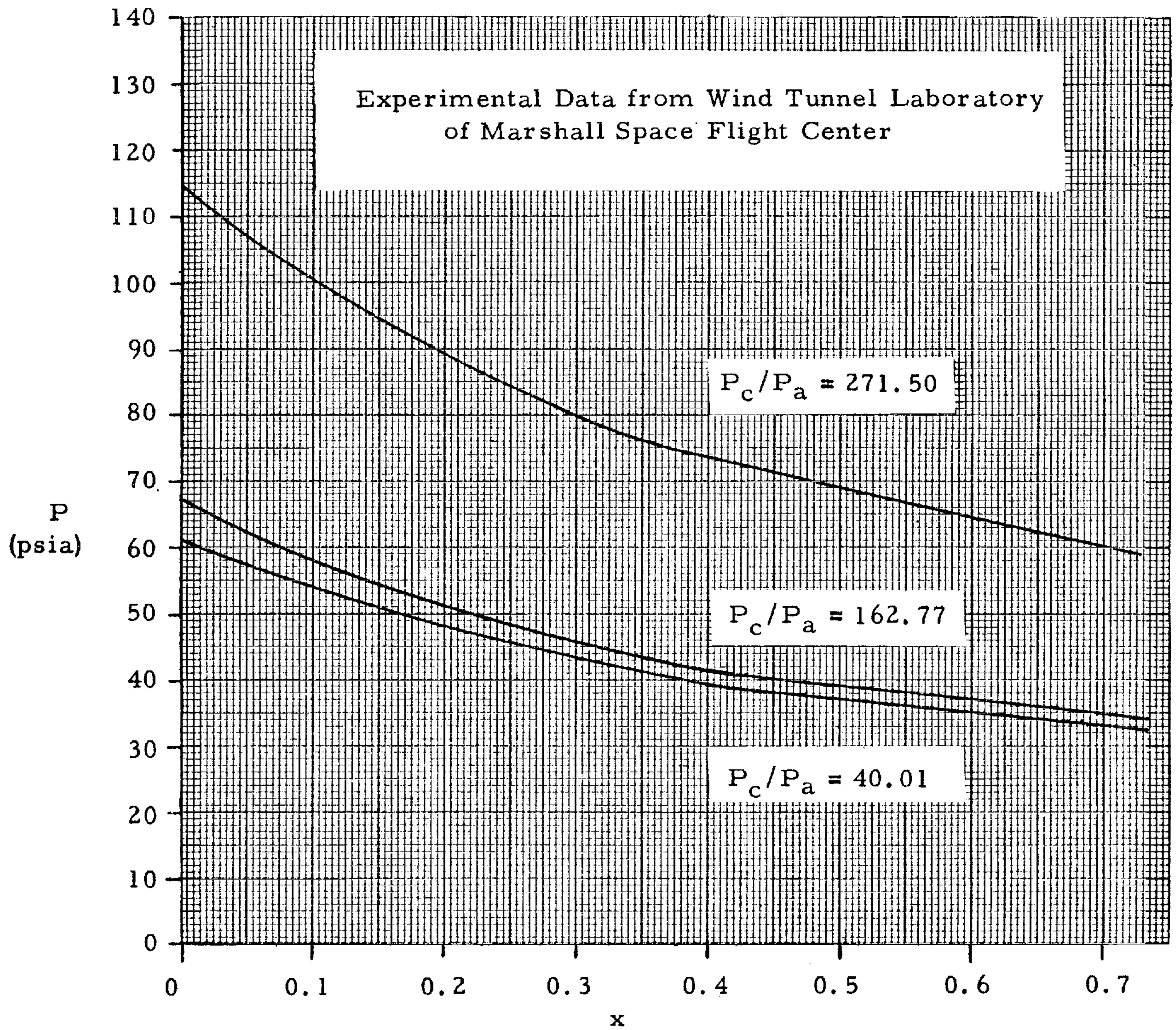


Figure 4 - Pressure Distribution Along the Wall of a Tsien Nozzle

monotonically from the beginning of the cone to the end of the cone; therefore, boundary layer separation should not occur in the typical Tsien nozzle.

(2) In the derivation of the thrust equation (2-11), isentropic flow is assumed in the nozzle. The flow in the nozzle is approximately adiabatic and since the nozzle is short, the frictional effects are comparatively small; therefore, the flow may be considered isentropic. Equation 2-11 indicates that the thrust coefficient for the Tsien nozzle is a function of the ratio of specific heats and the pressure ratio, P_c/P_a. By assuming a value of the ratio of specific heats, the theoretical thrust coefficient can be plotted versus pressure ratio as shown in Figure 5. The wind tunnel test results are also shown in the same figure so that a comparison of theoretical and experimental data can be made. The results show that the experimental thrust coefficient are very close.

From the theoretical analysis and experimental tests, it can be concluded that the Tsien method may be used to design subsonic or sonic nozzles and the performance of the Tsien nozzle gives a good agreement with the theoretical values.

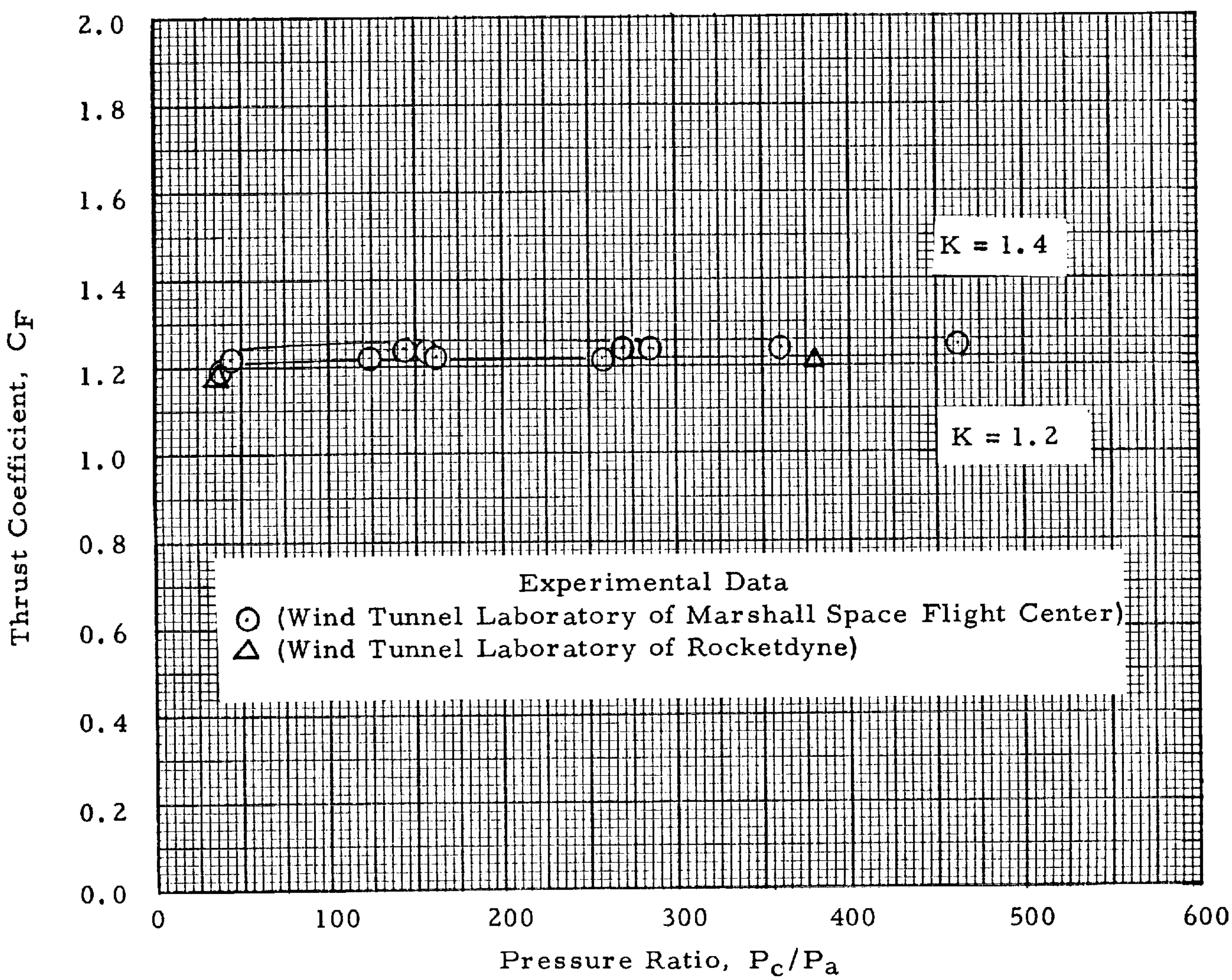


Figure 5 - Comparison of Theoretical and Experimental Thrust Coefficient with Different Pressure Ratios for a Tsien Nozzle

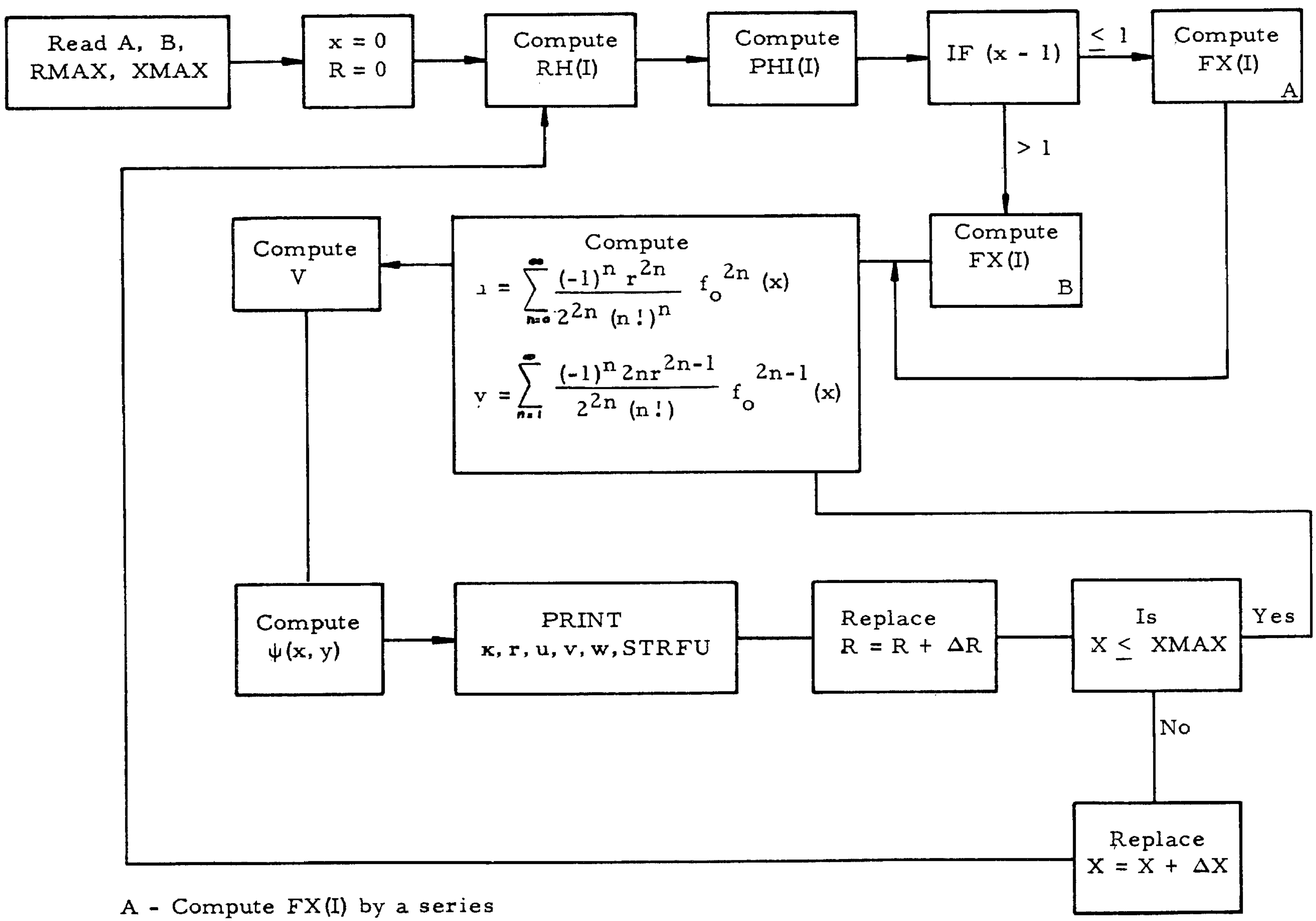
REFERENCES

1. Hsne-Shen Tsien, "On the Design of the Contraction Cone for a Wind Tunnel", Journal of the Aeronautical Sciences, vol. 10, pp. 68-70, February, 1943

2. Ascher H. Shapiro, The Dynamics and Thermodynamics of Compressible Fluid Flow, The Ronald Press Company, vol. 1, pp. 73-103

APPENDIX

FLOW CHART OF THE TSIEN NOZZLE DESIGN



A - Compute FX(I) by a series
B - Compute FX(I) by an exponential function

```
C     DESIGN OF THE CONTRACTION CONE FOR A WIND TUNNEL
      DIMENSION RH%20¤,PHI%20¤,FX%21¤
      READ1,A,B,RMAX,XMAX
    1 FORMAT%F10.5,F10.5,F10.5,F10.5¤
      PRINT103
  103 FORMAT%1H0,1HX,9X,1HR,9X,1HU,9X,1HV,9X,3HVEL,7X,5HSTRFU¤
      X#0.0
  102 R#0.0
      STP#SQRTF%2.*3.1416¤
      PH #%1./STP¤*%EXPF%-0.5*%X**2¤¤¤
      RN#3.0
      RH%1¤#X
      RH%2¤#X*X-1.
      DO2I#3,20
      RH%I¤#X*RH%I-1¤-RN*RH%I-2¤
    2 RN#RN&1.
      RN#-1.0
      DO3I#1,20
      PHI%I¤#RN*PH*RH%I¤
    3 RN#RN*%-1.¤
      Z#X
      IF%Z-1.¤5,5,15
    5 Z2#Z*Z
      Z3#Z2*Z
      Z5#Z3*Z2
      Z7#Z5*Z2
      Z9#Z7*Z2
      Z11#Z9*Z2
      ERFX#0.56419*%Z-Z3/3.&Z5/10.-Z7/42.&Z9/216.-Z11/1320.¤
      FX%1¤#A&B*ERFX
      GO TO 6
   15 Y#1./%2.*Z*Z¤
      Y1#9.*Y/%1.&10.*Y¤
      Y2#8.*Y/%1.&Y1¤
      Y3#7.*Y/%1.&Y2¤
      Y4#6.*Y/%1.&Y3¤
      Y5#5.*Y/%1.&Y4¤
      Y6#4.*Y/%1.&Y5¤
      Y7#3.*Y/%1.&Y6¤
      Y8#2.*Y/%1.&Y7¤
      Y9#Y/%1.&Y8¤
      SER#1./%1.&Y9¤
      ERFX#.5-SER/%3.5449*Z*EXPF%Z*Z¤¤
      FX%1¤#A&B*ERFX
    6 CONTINUE
      DO7I#1,20
      K#I&1
    7 FX%K¤#B*PHI%I¤
  101 SUMU#0.0
      C#-1.
      P#1.
      W#1.
      R2#R*R
      T#4.
```

```
      DO8I#2,20,2
      SUMU#SUMU&%C*R2*FX%I□□/%T*%W**2□□
      R2#R2*R*R
      T#T*4.
      C#C*%-1.□
      P#P&1.
    8 W#W*P
      U#FX%1□&SUMU
      SUMV#0.0
      C#&1.
      P#2.
      W#2.
      T#16.
      Q#R*R*R
      D#2.
      DO9I#3,21,2
      SUMV#SUMV&%C*2.*D*Q*FX%I□□/%T*%W**2□□
      C#C*%-1.□
      D#D&1.
      R2#R*R
      Q#Q*R2
      T#T*4.
      P#P&1.
    9 W#W*P
      PHBR#PH*B*%-.5□*R
      V#PHBR&SUMV
      VEL#SQRTF%U**2&V**2□
      SUMST#0.0
      C#-1.
      P#1.
      T#4.
      W#1.0
      QG#4.
      R4#R*R*R*R
      DO10I#2,20,2
      ZUMI#R4/QG
      SUMST#SUMST&%C*FX%I□*ZUMI/%T*%W**2□□□
      C#C*%-1.□
      P#P&1.
      W#W*P
      QG#QG&2.0
      T#T*4.
   10 R4#R4*R*R
      STRFU#0.5*R*R*FX%1□&SUMST
      PRINT11,X,R,U,V,VEL,STRFU
   11 FORMAT%F10.6,F10.6,F10.6,F10.6,F10.6,F20.10□
      R#R&.01
      IF%R-RMAX□101,101,12
   12 X#X&0.1
      IF%X-XMAX□102,102,13
   13 PRINT 14
   14 FORMAT% 10X, 22HPROCESS IS COMPLETE   □
      END
```

```
C     DETERMINE THE THRUST COEFFICIENT IN A SIMPLE CONVERGING NOZZLE
      READ1,VK1,VK2,DVK
      READ1,PR1,PR2,DPR
    1 FORMAT%3F10.5¤
      STP#PR1
    2 A#1./%VK1-1.¤
      B#%2./%VK1&1.¤¤**A
    3 COEFF#2.*B-1./PR1
      PRINT4,VK1,PR1,COEFF
    4 FORMAT%3F10.5¤
      PR1#PR1&DPR
      IF%PR1-PR2¤3,3,5
    5 VK1#VK1&DVK
      IF%VK1-VK2¤8,8,6
    8 PR1#STP
      GOTO2
    6 PRINT7
    7 FORMAT% 10X,22HPROCESS IS COMPLETE    ¤
      END
```